# The Galor-Weil Model Revisited: A Quantitative Exercise

Nils-Petter Lagerlöf*

Department of Economics

York University

4700 Keele St.

Toronto ON Canada M3J 1P3

e-mail: lagerlof@econ.yorku.ca

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**Abstract:**

The long-run growth model of Galor and Weil (AER 2000) is examined quantitatively. We first give parametric forms to some functions which were only given on general form in the original article. We then choose numerical parameter values in line with calibrations of related long-run growth models, and with data. Finally, we simulate the model. We find, *inter alia*, that the time paths for population, and other variables, display oscillatory behavior: they move in endogenous cycles. As the economy transits from Malthusian stagnation to modern growth these oscillations die out. This is consistent with population growth rates fluctuating considerably in historical data, but having stabilized in modern economies. We also show that these cycles are *not* an artifact of the two-period life setting: allowing adults to live on after the second period of life with some probability does not make the oscillations go away. Rather, the cycles are driven by fertility being proportional to per-capita income minus the parental subsistence requirement. When population is large, and per-capita incomes close to subsistence, fertility is therefore sensitive to changes in population levels.

# 1 Introduction

A number of recent papers have modelled growth in the "very long run." Perhaps most cited is Galor and Weil (2000) [henceforth GW], who replicate a three-stage process of economic development.[1] An economy starting off in a state of Malthusian stagnation, endogenously enters first a phase of post-Malthusian growth (where population growth and technological progress increase simultaneously), and later a stage of modern growth (where population growth declines and technological progress spurts and stabilizes at a sustained positive rate). This is consistent with the broad facts about development in Western Europe over the last couple of millennia (cf Figure 1).

Even though their model is essentially about explaining time paths, GW's analysis is theoretical and qualitative; no attempt is made to simulate the model. A large number of models explaining similar facts have recently been examined quantitatively, but none that contains the many ingredients of the original GW model. Here we specify parametric forms for some previously implicit functions in the GW model, put reasonable numbers on the parameters and choose initial conditions, and then simulate the model. Overall we find that the GW model performs well quantitatively, in the sense that it can replicate the patterns in Figure 1.

We believe the value-added of this exercise is threefold. First, taking the GW model seriously from a quantitative perspective may bring new insights,

[1] See also Galor and Weil (1999) for an informal discussion and motivation of some of the themes in the GW model, and Galor (2005, Section 4) who presents the GW model with slightly different layout and notation.

and inspire new ways to think about long-run growth, both in terms of modelling and when simulating and calibrating models with related structures (e.g. Galor and Moav 2002). That may in turn help asses the empirical relevance of an expanding and celebrated, but so far predominantly theoretical, growth literature.

Second, since the GW model is so complex and multi-dimensional there is a pedagogical value of simulating it. GW use two-dimensional phase diagrams to illustrate what happens over time to four state variables simultaneously. In our simulations we can compare the time paths of each variable and see how it is driven by the model's assumptions, and how it relates to the paths of the other variables.

Third, our calibration exercise generates a result that GW themselves did not note, or emphasize: endogenous cyclical (oscillatory) behavior of population growth in the Malthusian regime, and the dying off of these cycles as the economy transits into sustained technological progress. The mechanics driving these cycles is Malthusian. The key feature is that fertility is proportional to per-capita income above the subsistence requirement of the parent. When population in the current period is large, land per agent is low, and per-capita income close to subsistence. Thus fertility is close to zero and population in the next period pushed almost to extinction. This makes next period's per-capita incomes high, spurring a phase of population growth until over-population sets in and the cycle starts all over again.

The model needs not generate oscillations. We show analytically (and illustrate in simulations) that if, for example, the fixed time cost of children is sufficiently (maybe unrealistically) high virtually all other results remain but the paths become non-oscillatory. At the same time, some extensions which we may intuitively think would rule out oscillations do in fact not: allowing adults to live for more than one period – a "perpetual youth" setting à la Yaari (1965) – cannot make the oscillations go away. In short, lower adult mortality only raises steady state population levels, and thus pushes per-capita incomes lower and closer to subsistence, amplifying the oscillatory features of the steady state.[2]

Whether or not these cycles are interesting is perhaps a matter of taste; we believe that our quantitative exercise should have a lot of value added also without them. But rather than trying to eliminate the oscillatory features of the GW model (or criticize them) we argue that they are interesting in their own right. There are two reasons for this. First, cyclical behavior in population growth is often encountered in historical data. Figure 2 shows the annualized population growth rate for Europe from 100 B.C. to modern times. As seen, this has fluctuated a great deal; note e.g. the Black Death in the 14th century. Such large movements in population growth do not seem to be present in the rich world today. We do not suggest that the Black Death should be interpreted literally as an endogenous cycle, but epidemics (and probably wars and famines too) tend to be more lethal in over-populated environments. Examining models with Malthusian backlashes may thus say something important about long-run population dynamics. Making the same

[2]One qualitative change to the model that *could* make the oscillations go away is allowing for property rights to land. Parents may then take into account the welfare effects on each child from diluting the family landholdings, and reduce fertility before over-population sets in. However, as we discuss in Section 4, this result is probably not robust to alternative ways of modelling property rights.

point another way: a model with endogenous cycles has an *endogenous propagation mechanism* for shocks were we to introduce them.

A second reason these oscillations are interesting is that they are driven by much the same mechanisms as some environmental degradation models, in particular the Easter Island disaster story of Brander and Taylor (1998). Our study may thus help shed light on how, if, and why, we have left a state where such environmental/Malthusian backlashes occur.

This paper proceeds as follows. The rest of this section gives some examples of earlier related work. Next, Section 2 briefly sums up the central equations and components of the Galor-Weil model. Subsection 2.1 then provides the functional forms we use, and Subsection 2.2 sets up the whole dynamical system. The baseline quantitative exercise follows in Subsection 2.3. Section 3 provides some sensitivity analysis; we find that allowing for a higher time cost of children will make the oscillations vanish, but a perpetual youth setting will not. Section 4 concludes.

## 1.1 Earlier work

Other attempts to explain long-run growth patterns quantitatively include Hansen and Prescott (2002). Some of their results are driven by a postulated, but not derived, hump-shaped pattern over time of population growth, as we see in the data (cf Figure 1). GW generate such a hump-shape endogenously, through agents' fertility choices. The Hansen-Prescott study is useful for comparing our results, and also provides useful guidance when we choose parameter values.[3]

[3]For other applications of the Hansen-Prescott model, see e.g. Ngai (2004).

Some long-run growth models with endogenous demographics share many, but never all, of the ingredients in the GW model. See e.g. Boldrin and Jones (2002), Doepke (2004), Fernandez-Villaverde (2001), Jones (2001), Kögel and Prskawetz (2001), Lagerlöf (2003c), and Tamura (1996, 2002, 2004). For example, these all lack any direct erosion effect on human capital coming from technological progress, and the associated link from technological progress to educational choices.

A long-run growth model replicating a pattern of declining volatility in population growth is set up by Lagerlöf (2003a,b). There, however, population growth fluctuates not due to endogenous cycles, but exogenous mortality shocks (epidemics). An exogenous mortality function is postulated to make the model generate a pattern of declining mortality volatility. In our simulations of the GW model, by contrast, the declining volatility is a reflection of the changing stability features of the system itself as technology starts to grow, and the economy leaves the Malthusian trap.

Our study also adds to a large literature on endogenous and deterministic fluctuations in population growth and incomes, both in models and data; see e.g. Azariadis et al. (2004), Easterlin (1987), Greenwood et al. (2005, Appendix B), and Samuelson (1976). However, these papers do not focus directly on Malthusian cycles, as discussed here, or the dampening of such cycles with the transition to modern growth.

As mentioned already, Brander and Taylor (1998) model Malthusian cycles in population and natural resources in ways similar to GW, but without any transition from stagnation to modern growth. Our exercise may thus also add to a new and interesting environmental literature.

## 2 The Galor-Weil Model

We first give a very brief summary of the original GW model, before showing what we add to it terms of parametric forms and quantitative analysis. We refer to the original paper for details.

This is an overlapping-generations model, where agents live for two periods: as children and adults. In adulthood agents earn income, consume, decide how many children to rear, and invest in their children's education. Children are passive; they earn no income and consume nothing, but they receive an education by their parents.

The income per unit of time of an agent who is adult in period $t$ is denoted $z_t$, and given by

$$z_t = h_t^\alpha x_t^{1-\alpha} = h_t^\alpha \left( \frac{A_t X}{L_t} \right)^{1-\alpha}, \tag{1}$$

where $L_t$ is the total adult population, $X$ is total land (which is exogenous and constant), and $A_t$ is the level of technology (which is land augmenting). The product $A_t X$ is referred to as (total) effective resources, and $x_t = A_t X / L_t$ is thus effective resources per worker. $h_t$ is human capital of a period-$t$ adult, and $\alpha$ is the labor share in goods production.

Technology progresses from period $t$ to $t+1$ at rate

$$g_{t+1} = \frac{A_{t+1} - A_t}{A_t}. \tag{2}$$

Human capital is produced using

$$h_{t+1} = h(e_{t+1}, g_{t+1}), \tag{3}$$

where $e_{t+1}$ denotes education invested in children in period $t$ (who become adults in period $t+1$). Note that technological change affects human capital accumulation.

The following conditions are imposed on $h(\cdot)$:

$$\begin{array}{ll} h_e(e,g) > 0 & h_{ee}(e,g) < 0 \\ h_g(e,g) < 0 & h_{gg}(e,g) > 0 \end{array}. \tag{4}$$

The interpretation goes as follows. Education raises human capital, but with a declining marginal effect. Technological progress *reduces* human capital (making knowledge obsolete); this "erosion effect" is also declining on the margin.

GW also assume that

$$h_{eg}(e,g) > 0, \tag{5}$$

which implies that technological progress raises the *return* to investing in education; or, equivalently, that the erosion effect of technological change declines with eduction.

The utility function is given by

$$u_t = (1-\gamma)\ln c_t + \gamma \ln(n_t h_{t+1}), \tag{6}$$

where $\gamma \in (0,1)$, $c_t$ is consumption, and $n_t$ the number of (surviving) children.[4] The budget constraint for consumption is given by

$$c_t = z_t \left[1 - (\tau + e_{t+1})n_t\right]. \tag{7}$$

[4]GW do not explicitly model child mortality. However, unless children carry their fixed time and education costs *before* dying (quiet unlikely considering that the higher mortality rates historically were among infants), nothing would change if we allowed for mortality, except $n_t$ would then be fertility net of (infant) mortality.

Each unit of education costs one unit of time; $\tau$ is a fixed time cost so each child costs $(\tau + e_{t+1})$ units of time to rear.[5]

Utility in (6) is maximized subject to four constraints: the budget constraint in (7); the human capital production function in (3); a subsistence consumption constraint: $c_t \geq \widetilde{c}$; and a non-negativity constraint on education: $e_{t+1} \geq 0$.

The first-order conditions will look differently, depending on whether $c_t \geq \widetilde{c}$ and $e_{t+1} \geq 0$ are binding. The first-order condition for $n_t$ implies

$$n_t[\tau + e_{t+1}] = \begin{cases} \gamma & \text{if } z_t \geq \widetilde{z} \\ 1 - \frac{\widetilde{c}}{z_t} & \text{if } z_t \in (\widetilde{c}, \widetilde{z}) \\ 0 & \text{if } z_t \leq \widetilde{c} \end{cases}, \tag{8}$$

where $\widetilde{z} = \widetilde{c}/(1-\gamma)$. As long as $z_t \in (\widetilde{c}, \widetilde{z})$, the subsistence consumption constraint, $c_t \geq \widetilde{c}$, is binding, and total time spent with children is rising in $z_t$. When $z_t$ exceeds $\widetilde{z}$ time spent with children is constant at $\gamma$. When $z_t \leq \widetilde{c}$ fertility is zero, and the population dies out.

The first-order condition for $e_{t+1}$ gives

$$G(e_{t+1}, g_{t+1}) \begin{cases} = 0 & \text{if } e_{t+1} > 0 \\ > 0 & \text{if } e_{t+1} = 0 \end{cases}, \tag{9}$$

where

$$G(e_{t+1}, g_{t+1}) = (\tau + e_{t+1})\, h_e(e_{t+1}, g_{t+1}) - h(e_{t+1}, g_{t+1}). \tag{10}$$

[5] The original GW article uses an equivalent, but slightly more cumbersome, formulation, where $\tau^e$ denotes the time cost per unit of education, and $\tau^q$ a fixed time cost per child, so that total time cost per child becomes $\tau^q + \tau^e e_{t+1}$. Here we follow e.g. Galor (2005) and normalize $\tau^e$ to unity.

This defines education invested in children ($e_{t+1}$) as an implicit function of technological progress ($g_{t+1}$). The assumptions made about $h(e_{t+1}, g_{t+1})$ above imply that (as long as $e_{t+1} > 0$) $e_{t+1}$ is increasing in $g_{t+1}$:[6]

$$e'(g_{t+1}) = -\frac{G_g(e_{t+1}, g_{t+1})}{G_e(e_{t+1}, g_{t+1})} > 0. \tag{11}$$

Next GW assume that

$$G(0,0) = \tau h_e(0,0) - h(0,0) < 0, \tag{12}$$

which implies that there exists some $\widehat{g}$, such that $e_{t+1}$ is constrained to zero if $g_{t+1} < \widehat{g}$. We can thus write:

$$e(g_{t+1}) \begin{cases} > 0 & \text{if } g_{t+1} > \widehat{g} \\ = 0 & \text{if } g_{t+1} \leq \widehat{g} \end{cases}. \tag{13}$$

Technological progress from $t$ to $t+1$ is assumed to depend on the education of period-$t$ adults, and period-$t$ adult population size (a scale effect, of sorts):

$$g_{t+1} = g(e_t, L_t), \tag{14}$$

where $\partial g(e_t, L_t)/\partial e_t > 0$. It is assumed that the scale effect has an upper bound, i.e., $\lim_{L\to\infty} g(e; L)$ is finite. It is also assumed that

$$g(0, L_t) > 0, \tag{15}$$

[6] To see this, note that $h_{ee}(\cdot) < 0$ implies that

$$G_e(e_{t+1}, g_{t+1}) = (\tau + e_{t+1})\, h_{ee}(e_{t+1}, g_{t+1}) < 0.$$

Similarly, $h_g(\cdot) < 0$, and $h_{eg}(\cdot) > 0$, imply that

$$G_g(e_{t+1}, g_{t+1}) = (\tau + e_{t+1})\, h_{eg}(e_{t+1}, g_{t+1}) - h_g(e_{t+1}, g_{t+1}) > 0.$$

i.e., there is some (possibly very slow) technological progress also in absence of education.

## 2.1 Parametric forms

So far, we have presented the same setting as GW, where some functions are defined only implicitly. To be able to simulate the model we must find parametric functional forms for $h(e_{t+1}, g_{t+1})$ and $g(e_t, L_t)$. This is hard to come up with, but the following works, and seems intuitive:

$$h_{t+1} = h(e_{t+1}, g_{t+1}) = \frac{e_{t+1} + \rho\tau}{e_{t+1} + \rho\tau + g_{t+1}}, \tag{16}$$

where $\rho \in (0, 1)$ is exogenous and can be interpreted as a part of the fixed time cost, $\tau$, that helps build human capital, so that $e_{t+1} + \rho\tau$ can be thought of as effective education. That is, nursing and looking after small children help build their human capital, but not as effectively as education, since $\rho < 1$. (This formulation is borrowed from Lagerlöf 2003a.)

Applying the expression defining optimal education in (10) to the parametric form in (16), we can derive optimal $e_{t+1}$ as

$$e(g_{t+1}) = \max\left\{0, \sqrt{g_{t+1}\tau(1-\rho)} - \rho\tau\right\}. \tag{17}$$

It is easy to check that (16) satisfies the assumptions in (4). The assumption in (5) typically does *not* hold, but this is only a sufficient, not necessary, assumption to generate the result that $e_{t+1}$ is increasing in $g_{t+1}$ [which holds anyhow; see (17)].[7] It can also be seen that (12) holds, and that

[7] More precisely, using (16) it can be seen that

$$h_{eg}(e_{t+1}, g_{t+1}) = \frac{e_{t+1} + \rho\tau - g_{t+1}}{(e_{t+1} + \rho\tau + g_{t+1})^3}.$$

$\widehat{g} = \rho^2\tau/(1-\rho)$.

Next, let technological progress take the form:

$$g_{t+1} = g(e_t, L_t) = (e_t + \rho\tau)a(L_t), \tag{18}$$

where $a'(L_t) > 0$ (see below). In a fashion similar to (16) technological progress depends on effective education in period $t$, i.e. $e_t + \rho\tau$. Note also that $g_{t+1} > 0$ when $e_t = 0$. Choosing the functional forms as in (16) and (18) greatly simplifies the algebra later on.

The scale effect, $a(L_t)$, could take many functional forms. We choose:

$$a(L_t) = \min\{\theta L_t, a^*\} \tag{19}$$

where $\theta, a^* > 0$. Thus, population affects technological progress linearly for $L_t \leq a^*/\theta$, and then stays flat.[8]

#### 2.1.1 Education dynamics for fixed population

Using (18) and (17), we can write $e_{t+1}$ as a function of $e_t$, and $L_t$:

$$e_{t+1} = \max\left\{0, \sqrt{(e_t + \rho\tau)a(L_t)\tau(1-\rho)} - \rho\tau\right\} \equiv \phi(e_t, L_t). \tag{20}$$

Holding population, $L_t$, constant this constitutes a one-dimensional difference equation in $e_t$, the configuration of which depends on population size. Increasing $L_t$ – and thus $a(L_t)$ – the difference equation at some point switches

Inserting optimal $e_{t+1}$ from (17) this is seen to be positive only when $g_{t+1} < \tau(1-\rho)$, which does not hold in the modern growth regime in our baseline case.

[8]Since the economic content of the model will drive the interesting non-linearities of the derived dynamical system, we want to ensure that we are not sneaking any extra non-linearities in through the back door. Given the restrictions that $a'(L_t) > 0$, and $a(L_t)$ being bounded from above, it seems that (19) is a reasonable choice.

from having a unique steady state with zero education and slow growth, to having a unique steady state with faster growth and positive education.

The switch from zero to positive education occurs when $a(L_t)$ exceeds $\rho/(1-\rho)$, i.e., when population exceeds

$$\widehat{L} = \frac{\rho}{\theta(1-\rho)}. \tag{21}$$

[We are assuming $\rho/(1-\rho) < a^*$ so that the switch occurs before $a(L_t)$ has reached its maximum; cf (19).]

## 2.2 The full dynamical system

The full dynamics of this model are characterized by a non-linear four-dimensional system of difference equations. One can write this system in terms of the variables $x_t$, $g_t$, $e_t$, and $L_t$, where (recall) $x_t = A_t X/L_t$ is effective resources per worker; that approach is taken by Galor (2005).[9] Here we instead write the system in terms of $A_t$, $g_t$, $e_t$, and $L_t$. The two ways are in principle equivalent, but as we shall see, looking at the difference equation for $L_t$ while keeping $A_t$ (rather than $x_t$) constant helps us understand the oscillatory population dynamics in the Malthusian regime.

First use the human capital production function in (16) to write per worker income, $z_t$, given in (1), as a function of the four state variables $A_t$,

[9] In fact, the original GW article (different from Galor 2005) analyzed an approximate system, where $L_t$ was kept constant, thus making it three-dimensional and somewhat easier to visualize. To simulate the model we need to write down all four difference equations explicitly.

$g_t$, $e_t$, and $L_t$, i.e.,

$$z_t = \left[\frac{e_t + \rho\tau}{e_t + \rho\tau + g_t}\right]^{\alpha} \left[\frac{A_t X}{L_t}\right]^{1-\alpha} \equiv z(A_t, g_t, e_t, L_t). \tag{22}$$

Next, use the expression for fertility in (8), and the expression for education time in (20), to write

$$n_t = \left\{ \begin{array}{cc} \frac{\gamma}{\tau+\phi(e_t,L_t)} & \text{if } z(A_t, g_t, e_t, L_t) \geq \widetilde{z} \\ \\ \frac{1-\frac{\widetilde{c}}{z(A_t,g_t,e_t,L_t)}}{\tau+\phi(e_t,L_t)} & \text{if } z(A_t, g_t, e_t, L_t) \in (\widetilde{c}, \widetilde{z}) \\ \\ 0 & \text{if } z(A_t, g_t, e_t, L_t) \leq \widetilde{c} \end{array} \right\} \equiv \eta(A_t, g_t, e_t, L_t), \tag{23}$$

where (recall) $\widetilde{z} = \widetilde{c}/(1-\gamma)$.

We can now write the full system as

$$\begin{array}{r} A_{t+1} = [1 + g(e_t, L_t)]A_t \\ g_{t+1} = g(e_t, L_t) \\ e_{t+1} = \phi(e_t, L_t) \\ L_{t+1} = \eta(A_t, g_t, e_t, L_t)L_t \end{array}. \tag{24}$$

Note that we have parametric expressions for everything in this system, from using (18), (19), (20), (22), and (23). Putting numbers on these parameters, and choosing a set of initial values for the state variables ($A_0$, $g_0$, $e_0$, and $L_0$), it is in principle straightforward to simulate the model.[10] Before doing that, however, we want to give some intuition behind an interesting result which will show up in the simulations in the Malthusian phase of development: the oscillatory growth pattern of population.

[10]The algorithm looks as follows. Given parameter values, and an initial state vector, $(A_0, g_0, e_0, L_0)$, we use (24) to compute the state vector in the next period: $(A_1, g_1, e_1, L_1)$. Using $(A_1, g_1, e_1, L_1)$ as new inputs in (24) we then calculate $(A_2, g_2, e_2, L_2)$; and so on.

### 2.2.1 Reduced form population dynamics

Consider an economy situated in a Malthusian regime, where education time is constrained to zero. We can approximate the population dynamics in this economy by keeping the other state variables ($e_t$, $g_t$, and $A_t$) constant. Letting $A_t$ be constant implies that $g_t = 0$. Together with $e_t = 0$, this implies that $h_t = h(e_t, g_t)$ is constant. We can then write the difference equation for $L_t$ defined by (23) and (24) as:

$$L_{t+1} = \begin{cases} \left(\frac{\gamma}{\tau}\right) L_t & \text{if } L_t \leq \widetilde{L} \\ \left(\frac{1}{\tau}\right) \left[1 - \Omega L_t^{1-\alpha}\right] L_t \equiv \Psi(L_t) & \text{if } L_t \in (\widetilde{L}, \widetilde{\widetilde{L}}) \\ 0 & \text{if } L_t \geq \widetilde{\widetilde{L}} \end{cases} . \tag{25}$$

where

$$\begin{aligned} \Omega &= \tfrac{\widetilde{c}}{h^{\alpha}[XA]^{1-\alpha}} \\ \widetilde{L} &= \left(\tfrac{1-\gamma}{\Omega}\right)^{\frac{1}{1-\alpha}} \\ \widetilde{\widetilde{L}} &= \left(\tfrac{1}{\Omega}\right)^{\frac{1}{1-\alpha}} \end{aligned} . \tag{26}$$

This follows directly from (8). Using $z_t = [h]^{\alpha} [AX/L_t]^{1-\alpha}$, and setting $e_{t+1} = 0$, we see that $z_t \geq \widetilde{z}$ amounts to $L_t \leq \widetilde{L}$. Likewise, $z_t \in (\widetilde{c}, \widetilde{z})$ amounts to $L_t \in (\widetilde{L}, \widetilde{\widetilde{L}})$; and $z_t \leq \widetilde{c}$ amounts to $L_t \geq \widetilde{\widetilde{L}}$.

The difference equation in (25) can be illustrated in a simple 45-degree diagram, as shown in Figure 3. Given that $\gamma > \tau$, the function lies above the 45-degree line for $L_t \leq \widetilde{L}$. The steady state is given by the intersection of $\Psi(L_t)$ with the 45-degree line, which may be with a negative or positive slope. Appendix A shows that the slope is positive if

$$\tau > \frac{1-\alpha}{2-\alpha} \equiv \tau^{\text{osc}}, \tag{27}$$

in which case the steady state is non-oscillatory; if the inequality is reversed the steady state is oscillatory.[11] Figure 3 shows an oscillatory steady state (parameters are set as in the baseline case in Table 1; see below). As illustrated, starting off outside the steady state population levels will rise and fall in chaotic and complicated patterns.

These cycles have a Malthusian character. In economies with small populations ($L_t \leq \widetilde{L}$) effective resources are abundant, so per-worker income is high ($z_t \geq \widetilde{z}$), and fertility is at its maximum level, $\gamma/\tau$. As population expands, and effective resources per worker are depleted, a Malthusian backlash arrives when resource scarcity pushes fertility below replacement and population falls in levels from one period to the next (see Figure 3). What drives the oscillations is the feature that fertility is determined by the residual of per-capita income, after subsistence consumption is covered. Thus, there is always some level of population where per-capita income cannot cover subsistence consumption, pushing fertility to zero and population to extinction in the next period.

From Figure 4 we can understand intuitively what will happen when simulating the full system, where $A_t$, $e_t$, and $g_t$ evolve endogenously. Note that $A_t$ is always growing, albeit slowly when $L_t$ is low [see (18)]. In terms of Figure 4, an increase in $A_t$ shifts out the function mapping $L_t$ to $L_{t+1}$, thus raising the ("quasi") steady state population size. The economy looks as if it "chases" its steady state. Things suddenly change when population comes to exceed $\widehat{L}$, defined in (21). This is the threshold above which a quality-

[11] If the slope is between $-1$ and 0, the steady state is oscillatory but (locally) stable; if the slope is less than $-1$ the steady state is oscillatory and unstable.

quantity substitution sets in, making the population growth rate drop.

## 2.3 Simulating the full system: the baseline case

### 2.3.1 Parameter values

The numerical values in the baseline case are given in Table 1, together with brief explanations for all parameters. Below follows some motivation for how the values are chosen.

We let each period correspond to 20 years, corresponding to the length of one generation (cf e.g. Boldrin and Jones 2002).

The labor share of output, $\alpha$, is set to 0.6, which is the same as in e.g. Hansen and Prescott (2002, Table 3).

The fixed time cost, $\tau$, is set to 0.15. We may interpret this as each child carrying fixed costs (i.e., excluding education) of about 15% of the parent's income. This is in line with Haveman and Wolfe (1995) who estimate total expenditures on children in the U.S. in 1992 to 14.5% of GDP. The main components are the opportunity cost of the mother's time (but not the father's), and direct costs such as clothing and food. Other items, such as elementary and secondary education, are also included, although they would perhaps better belong to education time, $e_t$, rather than $\tau$. (Higher education is not included, though.) Excluding these items total child expenditure would fall to a little over 10% of GDP. In that sense, 0.15 may be slightly too high a number for $\tau$. However, a lower $\tau$ creates bigger oscillations in population; with $\tau$ somewhere below 0.135 population at some point becomes extinct. With $\tau$ at 0.15 we get some oscillations without population becoming extinct.

It could thus be a reasonable compromise.[12]

The gross population growth rate (i.e., fertility) in the modern growth regime, which we denote $n^*$, is set to one, meaning that population converges to a constant in levels under modern growth.

The rate of technological progress in the modern growth regime is set to fit the growth rate of per-worker income, $z_t$, to 2.4% per annum (cf Hansen and Prescott 2002, Table 1). With constant population size, education, and technological progress, using the goods production function in (22), we see that the growth rate in $z_t$ equals $(1 - \alpha) = 0.4$, times the growth rate of $A_t$, which we denote $g^*$. This implies a value for $g^*$ of about 2.36.

Education time in the modern growth regime, which we denote $e^*$, can be interpreted as spending on education in the modern growth regime as a fraction of GDP, which is about 7.5% in the U.S.A. today [see de la Croix and Doepke (2004)]. We thus set $e^* = 0.075$.

Next we derive expressions for $g^*$ and $e^*$ in terms of the exogenous parameters of the model. To this end we use (17) and (18), recalling that $a(L_t)$ approaches its maximum level, $a^*$, in the modern growth regime. This gives

$$\begin{array}{c} g^* = (a^*)^2 \tau(1-\rho) \\ e^* = a^*\tau(1-\rho) - \rho\tau \end{array} . \tag{28}$$

Given the target values for $e^*$, $g^*$, and $\tau$ above we can compute what the exogenous parameters $a^*$ and $\rho$ must be; the resulting values are shown in

[12]One could also get closer to 15% if using a higher estimate of the mother's opportunity cost of having children, as discussed by Haveman and Wolfe (1995, Footnote 2).

Table 1.[13]

We then set $\gamma$ so that population is constant in the modern growth regime. Using (8), we see that under modern growth $n^* = \gamma/(\tau + e^*)$. Given $n^* = 1$ and $e^*$ and $\tau$ chosen as above, this gives $\gamma \approx 0.225$.

The remaining parameters ($\theta$, $X$, and $\widetilde{c}$) are all normalized to one. These are neutral in the sense that after calibrating initial conditions (see below), they play no role.

### 2.3.2 Initial conditions

The initial population level, $L_0$, is set to a fraction of $\widehat{L}$ as given by (21). Recall that $\widehat{L}$ is the level of population above which education time becomes operative and growth spurts. The lower we set $L_0$, the longer it takes before an industrial revolution sets in, but the shapes of the growth paths are not affected once it arrives. We set $L_0 = 0.05\widehat{L}$. This suffices to get about 30 generations (about 600 years) of Malthusian stagnation.

We set initial technology, $A_0$, so that population is constant in the first period, i.e., we drop the economy off in the "quasi" steady state it would stay in if technology was constant (cf Figure 3). Given that initial income, $z_0$, falls between $\widetilde{c}$ and $\widetilde{z}$ (which will soon be seen to hold), from (8) we see that initial fertility is given by $n_0 = (1/\tau)\left[1 - \widetilde{c}/z_0\right]$. Setting this equal to one, we get

[13] We can use (28) to solve for $\rho$ in terms of $e^*$, $g^*$, and $\tau$:

$$\rho = -\frac{g^* + 2e^*}{2\tau} + \sqrt{\left(\frac{g^* + 2e^*}{2\tau}\right)^2 - \left(\frac{[e^*]^2 - \tau g^*}{\tau^2}\right)},$$

(noting that $\rho$ must be positive, thus disregarding the negative root); $a^*$ equals $g^*/(e^* + \rho\tau)$.

$z_0 = \widetilde{c}/(1-\tau)$.[14] Using (16) to (19) we see that we can write $h_0 = 1/[1+\theta L_0]$. The expression for $z_t$ in (1) gives $z_0 = [1/(1+\theta L_0)]^{\alpha}[XA_0/L_0]^{1-\alpha}$, which is set equal to the desired value for $z_0$. Given the choice of $L_0$, this gives $A_0$ as in Table 1.

Initial education, $e_0$, is set to zero, and initial technological change, $g_0$, is set to $\rho\tau\theta L_0$ [cf (18) and (19)]. Given the values of $\rho$, $\tau$, $\theta$, and $L_0$ this gives $g_0$ as in Table 1.

#### 2.3.3 The simulations

Figures 5 to 9 show the simulated time paths for some of the more important variables. The time paths become highly oscillatory, so to distinguish the broad movements from the shorter cycles we show the moving-average values over 5 periods (about 100 years), centered on the mid-period. Such smoothing could perhaps also give an idea about how the paths would look if using a setting with more realistic demographics (for example, agents could live for 15 periods of 5 years and bear children over 5 of these periods). It may also make the results more comparable to the experience of the whole of (Western) Europe, where different regions' cycles may not have been synchronized.

Figure 5 shows growth rates and education. Population growth oscillates strongly (despite being averaged out over 5 periods), which matches what we see in Figure 2. Growth in per-worker income fluctuates with population through the dilution effect.

The overall pattern is consistent with the three-stage process that Galor

[14] Note that $z_0 = \widetilde{c}/(1-\tau) < \widetilde{z} = \widetilde{c}/(1-\gamma)$, since $\tau < \gamma$.

and Weil described; notably the growth rates of technology and population rise simultaneously through a so-called post-Malthusian phase, before diverging. Note also, however, that per-worker income grows quite slowly long after technology growth has started to climb, due to the dilution effect from rising population. Income growth converges to 2.4% per year, as we have calibrated it.

Figure 6 shows the levels of technology, population, and effective resources per adult ($x_t = A_t X / L_t$). Note how $x_t$ and $L_t$ move in opposite directions until $L_t$ stabilizes in levels, and $x_t$ takes off.

Figure 7 shows where population size reaches the threshold level, $\widehat{L}$, as given by (21), after which point education starts rising. This coincides with a decline in population growth, and a spurt in technological progress in the other diagrams.

Figure 8 shows the levels of consumption and per-worker income. The fluctuations reflect those in effective resources per worker, and thus population; cf Figure 5.

Figure 9 shows how the volatility of population growth becomes quite large if not averaging over 5 periods. This can also be compared to the volatility in population levels in Figure 5; a relatively small fall in population levels can mean a large drop in population growth.

Note how population growth becomes more volatile in the beginning, before stabilizing in the modern growth regime. Intuitively, with very slow changes in technology population stays close to the ("quasi") steady state where we dropped it off when choosing the initial conditions. As population levels rise, technological progress accelerates. In terms of the reduced form population dynamics shown in Figure 4 the steady state starts moving to the right as technology levels increase. When "chasing" the steady state, the economy's population dynamics become more volatile.

# 3 Sensitivity analysis

## 3.1 A higher $\tau$

To generate time paths without oscillations we can set the time cost of children, $\tau$, equal to (or above) its oscillations threshold, $\tau^{\text{osc}}$, as given by (27). Given the baseline choice of $\alpha = 0.6$, we raise $\tau$ from 0.15 to $(1-\alpha)/(2-\alpha) \approx 0.28$. We then recalibrate the values for $\rho$, $a^*$, and $\gamma$, as well as the initial conditions $L_0$ and $A_0$. These are summed up in Table 2.[15]

The simulated paths for growth rates and levels for some variables are shown in Figures 10 and 11. As seen, they are clearly smoother than in the baseline case, although there is a small oscillation in population growth over the first couple of periods (having to do with the threshold being calculated from a reduced-form setting; note also that we are not smoothing the paths). The peak population growth rate is lower, since the maximum fertility rate, $\gamma/\tau$, is lower due to the higher $\tau$. Little else differs from the baseline case. Notably, the timing of the take-off is not altered visibly. The reason is that we have calibrated initial population, $L_0$, to the same distance from threshold

[15] The procedure is the same as in the baseline case. Given the baseline values for $e^*$ and $g^*$, and the new value for $\tau$, we compute $a^*$ and $\rho$ using (28). We choose $\gamma$ to set $n^* = \gamma/(\tau + e^*) = 1$. We set $L_0$ to 5% of the resulting new level of $\widehat{L}$ [see (21)]; and $A_0$ so that initial fertility, $n_0$, equals 1.

population level, $\widehat{L}$, at which the quality-quantity substitution sets in [see (21)].

It is not easy to say how realistic this value of $\tau$ is. If we interpret $\tau$ literally as a fixed time cost carried by each child, 28% of the parents' time endowment may seem extremely high, far above e.g. the 15% estimate of Haveman and Wolfe (1995). (They include much more than the pure time cost, so 15% should constitute an upper bound.) However, since the fraction $\rho$ of the time cost effectively constitutes a form of human capital investment in the child, a more liberal interpretation would suggest that the *pure* fixed time cost corresponds to $(1-\rho)\tau$. Given the recalibrated value for $\rho$ (see Table 2), one can see that $(1-\rho)\tau \approx 0.04$. This is close to the numbers used in some calibrations of related models, such as Echevarria and Merlo (1999).

## 3.2 Perpetual youth

One might conjecture that the two-period life structure is what drives the oscillations – it is not. To see this, now let agents die after the second, and later, periods of life with some exogenous probability, $m < 1$. (We abstract from child mortality; or, rather, we may think of $n_t$ as representing the number of surviving children.) This setting is essentially a "perpetual youth" model à la Yaari (1965).[16]

The only difference to the two-period life setting is that population now evolves according to: $L_{t+1} = L_t n_t + (1-m)L_t$, where $(1-m)L_t$ is the number of adults surviving for another period of life. The full dynamical system is

[16] The term perpetual youth is borrowed from Blanchard and Fischer (1989, Section 3.3). It stems from the probability of death being independent of age.

thus identical to that in (24), except for the last row where we add $(1-m)L_t$ to the right-hand side.[17] Setting $m = 1$ brings us back to the two-period life setting.

### 3.2.1 Reduced form population dynamics

To get an intuitive idea of how the dynamics change, consider first what happens in the reduced form case, where $A_t$, $g_t$, and $e_t$ are constant. As shown graphically in Figure 12, and is also seen analytically from (30) in Appendix A, lowering $m$ while holding fixed all other parameters at their baseline values shifts the function for $L_{t+1}$ upwards, at a given $L_t$. However, it still intersects the 45-degree line with a negative slope; the steady states are oscillatory also when $m < 1$.

Intuitively, lower mortality means larger population in steady state. More people today means less food, and thus lower fertility, and – if fertility falls low enough – fewer people tomorrow than today. Put another way, larger population means higher "Malthusian pressure," which is what drives the oscillations.

In a sense, the perpetual youth setting is *more* likely to show oscillatory population dynamics than the two-period life setting. As shown in Appendix A, in the perpetual youth setting the condition for the steady state to be non-

[17] We implicitly assume that those adults who live to the next period costlessly update their human capital to the level of the next generation. To relax this assumption we would have to keep track of a large set of heterogenous cohorts with different levels of human capital.

oscillatory – analogous to that in (27) – becomes:

$$\tau > \frac{1-\alpha}{1+m(1-\alpha)} \equiv \tau^{\text{osc}}, \tag{29}$$

meaning that $\tau^{\text{osc}}$ is *falling* in $m$; a lower $m$ means a higher $\tau^{\text{osc}}$. Put another way, if $\alpha$ and $\tau$ are such that the steady state is oscillatory ($\tau < \tau^{\text{osc}}$) in the baseline setting (where $m = 1$), that will hold also in the perpetual youth setting (where $m < 1$).

#### 3.2.2 Recalibrating the parameters

This reasoning holds when keeping constant all other parameters at their baseline values. Whether it holds also when recalibrating the model depends on if, and how, the critical parameters, $\alpha$ and $\tau$, change in the recalibrations.

The required recalibrations refer to the population growth rates; to be able to compare the paths to that of the baseline case we must make population constant under modern growth and in the initial period. Constant population under modern growth now implies that $n^* = \gamma/(\tau + e^*) = m$, which we target by recalibrating $\gamma$.[18] Population being constant in the first period means that $n_0 = (1/\tau)\left[1 - \widetilde{c}/z_0\right] = m$. Having set $L_0$ equal to 5% of $\widehat{L}$, as in the baseline case, this gives a new value for $A_0$. (The procedure is identical to that in the baseline case; see above.) The changes are summarized in Table 2, for $m = 0.5$ and $m = 0.25$. Crucially, $\alpha$ and $\tau$ are not recalibrated.

Figure 13 shows the reduced form population dynamics for $m = 0.5$ and $m = 1$. Notably, the intersection with the 45-degree line occurs with a

[18] We could alternatively adjust $\tau$. As shown in Section 3.1, this could make the oscillations go away regardless of $m$, so we do not consider that alteration here.

steeper slope when $m = 0.5$, meaning that the perpetual youth steady state is (locally) slightly more oscillatory than that of the two-period life setting.

Figure 14 shows the simulated paths for population growth rates in two cases, $m = 0.5$ and $m = 0.25$, together with the path from the baseline two-period life setting, $m = 1$, identical to that in Figure 9. Over the first 15-20 periods, or so, the oscillations are greater in the perpetual youth settings, consistent with the local properties of the steady states shown in Figure 13. But the perpetual-youth paths are less volatile later. This is driven by the lower recalibrated value for $\gamma$, implying lower levels for the maximum fertility rate, $\gamma/\tau$. Population thus expands at a slower rate when resources are abundant, prolonging the stable phase before a Malthusian backlash sets in. As a corollary, the hump-shaped time paths of the population growth rate become less visible when $m$ is lower. For $m = 0.25$ the path is flat for over ten generations before population growth declines.

In sum, even if a perpetual youth setting can mitigate the swings, this works only through the recalibration of the parameters, and it cannot make them go away completely. Moreover, the reduction in the oscillations comes at the cost of eliminating the hump-shaped population growth path, which was one of the central patterns that the GW model was designed to explain in the first place.

### 3.3 Alternative functional forms

At the cost of more complicated algebra, small variations on the functional forms chosen in (16) and (18) can make the difference equation in (20) exhibit multiple steady states. A zero-education steady state would then coexist with a high-education steady state at intermediate population levels, whereas there would be a unique zero-education steady state for low population levels and a unique high-education steady state for high population levels.[19]

Such a setting would probably generate much faster transition paths compared to the simulations shown here. Initial increases in population would have small (or no) effects as long as the economy stays in the zero-education steady state. As population comes to exceed a critical level and the configuration changes the zero-education steady state ceases to exist. At that point in time education, and thus also technological growth, quickly converge to the other steady state.

# 4 Conclusions

Among existing long-run growth models, the GW model is one of the more complex creations around, simply because it has so many ingredients. It has endogenous technological progress which depends both on population size and on educational levels; it has endogenous fertility decisions, which depend both on a quality-quantity choice (i.e., an education choice), and on whether a subsistence consumption constraint is binding, or not; it has human capital, which is increasing in education, but also being eroded through the process of technological progress; finally, it has land entering the production function, and being in fixed supply, so that population size has a negative effect on per worker income. The GW modelling approach is in a sense about explaining

[19]Such a multiple steady state configuration would correspond to how GW draw Figures 3 to 5 in their article.

"everything" in one unified framework.

Making a long story short, the GW model in the end produces a four-dimensional non-linear system of difference equations. In the original article, this is analyzed using two-dimensional phase diagrams where some of the variables are held constant. An alternative way to understand the model is to simulate it. To do this, we first specify functional forms, where GW used only implicit forms. We then set parameters and initial conditions, using common sense, facts, and by leaning on quantitative studies of other long-run growth models.

We then simulate the model. On the whole, the model is seen to replicate the growth paths we observe in data. To see this, compare the pattern in Figure 1 to the simulation in Figure 4; note in particular the hump-shaped pattern of population growth.

Somewhat more surprisingly, and not really explored by GW, the model is able to generate oscillatory cycles in population. This fits well with the patterns observed in historical data (see Figure 2 for Europe). Population expands when resources per capita are abundant, eventually generating a Malthusian backlash making population readjust downwards, after which the cycle starts all over again.

We can calibrate the model to do away with these oscillations. If, for example, the fixed time cost of children is sufficiently high, most results are unaltered but all paths become non-oscillatory. However, some extensions that may be expected to eliminate the oscillations do not do the trick. Allowing adults to live for more than one period only means higher population levels, thus pushing per-capita incomes closer to subsistence, amplifying the swings in population.

The GW model as presented here could be extended in many ways. There is a theoretically interesting link between the oscillations in the GW model, and those in Malthusian models with natural resource dynamics. Brander and Taylor (1998) model population cycles in a related context, where a human population consumes out of a finite, but renewable, natural resource. The natural resource regenerates itself over time, but is depleted by humans' harvests. Starting from a situation with abundant resources, population grows since food is plentiful. When population levels become sufficiently large, harvesting exceeds the natural regeneration rate, so the resource stock starts to decline. Population growth, however, depends on the *level* of the resource stock, and thus population keeps growing even as the resource stock is shrinking. This continues until the resource stock reaches a point where food scarcity sets in, and population starts to decline. The resource stock continues to fall until population density is small enough to allow the natural rate of regeneration to exceed the harvest rate, at which point the natural resource starts to grow again. To capture this in the GW model we could introduce a dynamic equation for $X_t$, which may now be interpreted as the natural resource base. The dynamical system would become five- instead of four-dimensional, but the simulation would follow the same algorithm.

Another, somewhat related, extension would be to allow for property rights in land. As the GW model is formulated, land is distributed equally so per-capita income depends on aggregate population. In an alternative setting where each family owns a plot of land of fixed size, and parents care about the income, or welfare, of each of their offspring, there would be an incentive for parents to reduce fertility in order not to dilute landholdings. This would presumably alter the over-population features of the model, and thus also the oscillations. On the other hand, to let per-capita income depend on aggregate population seems realistic for societies where land is not owned (hunter-gatherer societies), or if we think of $X$ as including other natural resources (e.g. fresh water).

At any rate, to simply *postulate* that each family owns a fixed plot of land seems at odds with the facts. In many historical contexts groups of people – clans, countries, etc. – have conquered new land when faced with land scarcity. Ideally, to take the idea of property rights seriously one should model them endogenously, and specify a technology for the appropriation of land, as in e.g. Grossman and Kim (1995). If population size is an input in land acquisition, that provides an extra motive for high fertility. In a symmetric equilibrium, where all clans/families are identical, the outcome may very well be over-population in the aggregate, with oscillations and Malthusian backlashes, not necessarily too different from what happens in the GW model as presented here.[20]

[20]See e.g. Grossman and Mendoza (2003) for a model where resource scarcity induces more appropriative competition.

# A Appendix

## A.1 Reduced form population dynamics in a perpetual youth setting

In the perpetual youth setting, the reduced form population dynamics are given by $L_{t+1} = L_t n_t + (1-m)L_t$, keeping $A_t$ constant and setting $e_t = g_t = 0$. Fertility, $n_t$, is given by (8), and $z_t$ by (1). Setting $e_{t+1} = 0$ we can then write:

$$L_{t+1} = \begin{cases} \left\{\frac{\gamma}{\tau} + 1 - m\right\} L_t & \text{if } L_t \leq \widetilde{L} \\ \left\{\left(\frac{1}{\tau}\right)\left[1 - \Omega L_t^{1-\alpha}\right] + 1 - m\right\} L_t \equiv \Psi(L_t) & \text{if } L_t \in (\widetilde{L}, \widetilde{\widetilde{L}}) \\ (1-m)\, L_t & \text{if } L_t \geq \widetilde{\widetilde{L}} \end{cases} \quad (30)$$

where the expressions for $\Omega$, $\widetilde{L}$, and $\widetilde{\widetilde{L}}$ are the same as in (26). Note that setting $m = 1$ brings us back to the original two-period life setting in (25).

### A.1.1 Conditions for a non-oscillatory steady state

Given that $\gamma > \tau$, from (30) we see that any (strictly positive) steady state level of $L_t$ – call it $\overline{L}$ – must lie on the $(\widetilde{L}, \widetilde{\widetilde{L}})$-interval, and be given by $\overline{L} = \Psi(\overline{L})$. Using (30) we see that

$$\overline{L} = \left[\frac{1 - \tau m}{\Omega}\right]^{\frac{1}{1-\alpha}}. \quad (31)$$

This steady state is non-oscillatory if $\Psi(L_t)$ intersects the 45-degree line with a positive slope, i.e., if $\Psi'(\overline{L}) > 0$, and vice versa the steady state is oscillatory if $\Psi'(\overline{L}) < 0$. Using (31), we see that $\Omega\overline{L}^{1-\alpha} = 1 - \tau m$, and using (30) we see that $\Psi'(\overline{L}) > (<)0$ amounts to

$$\begin{aligned} \Psi'(\overline{L}) &= \left(\tfrac{1}{\tau}\right)\left[1-(2-\alpha)\Omega\overline{L}^{1-\alpha}\right]+1-m \\ &= \left(\tfrac{1}{\tau}\right)\left[1-(2-\alpha)\left(1-\tau m\right)\right]+1-m > (<)0, \end{aligned} \tag{32}$$

which after some algebra gives (29).

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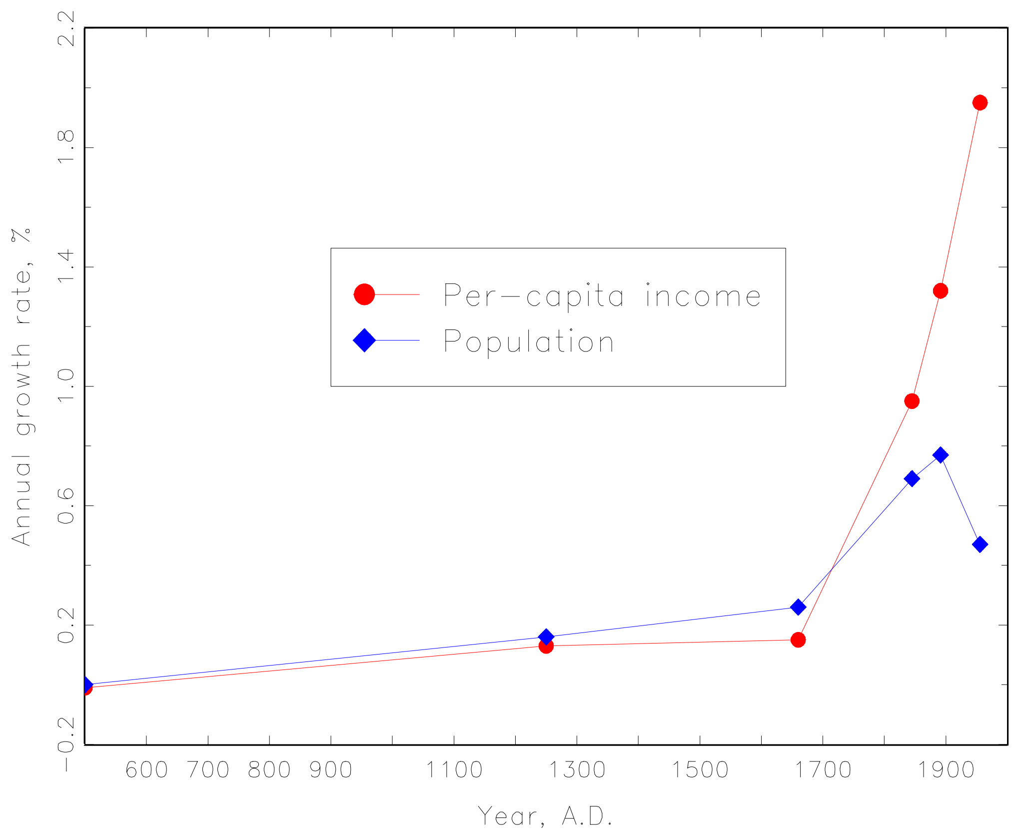


Figure 1: Growth rates in Western Europe. Calculated from Maddison (2003).

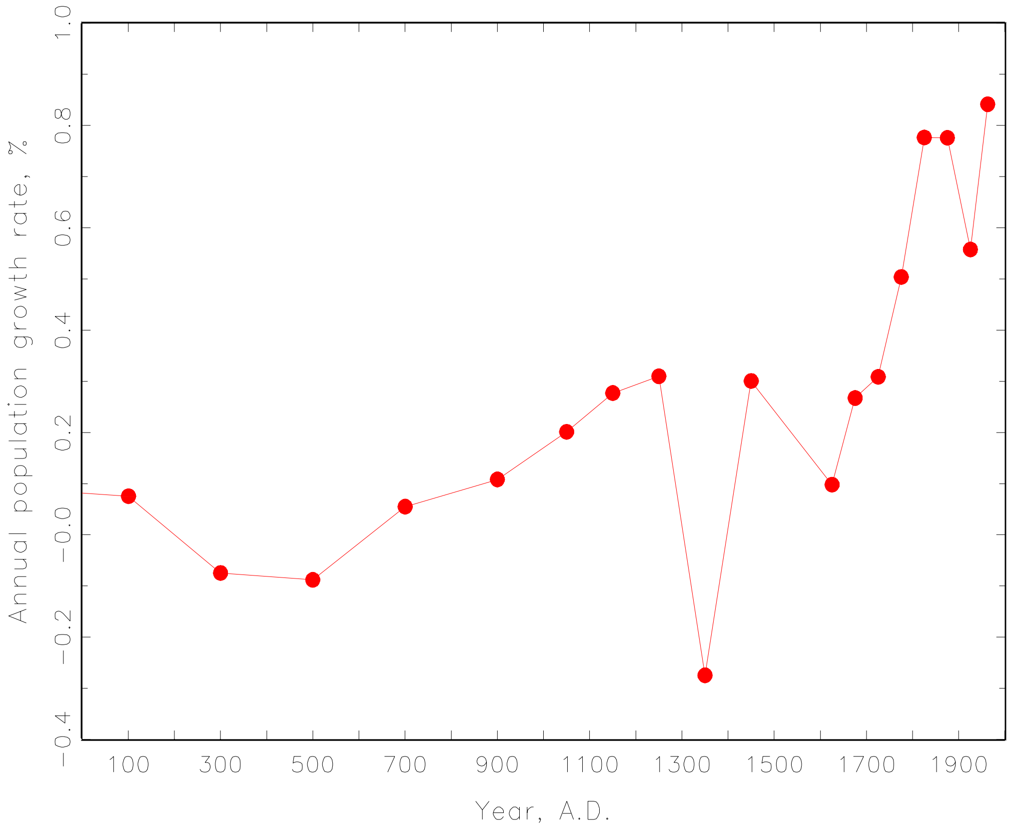


Figure 2: Population growth rate for Europe. Calculated from McEvedy and Jones (1978).

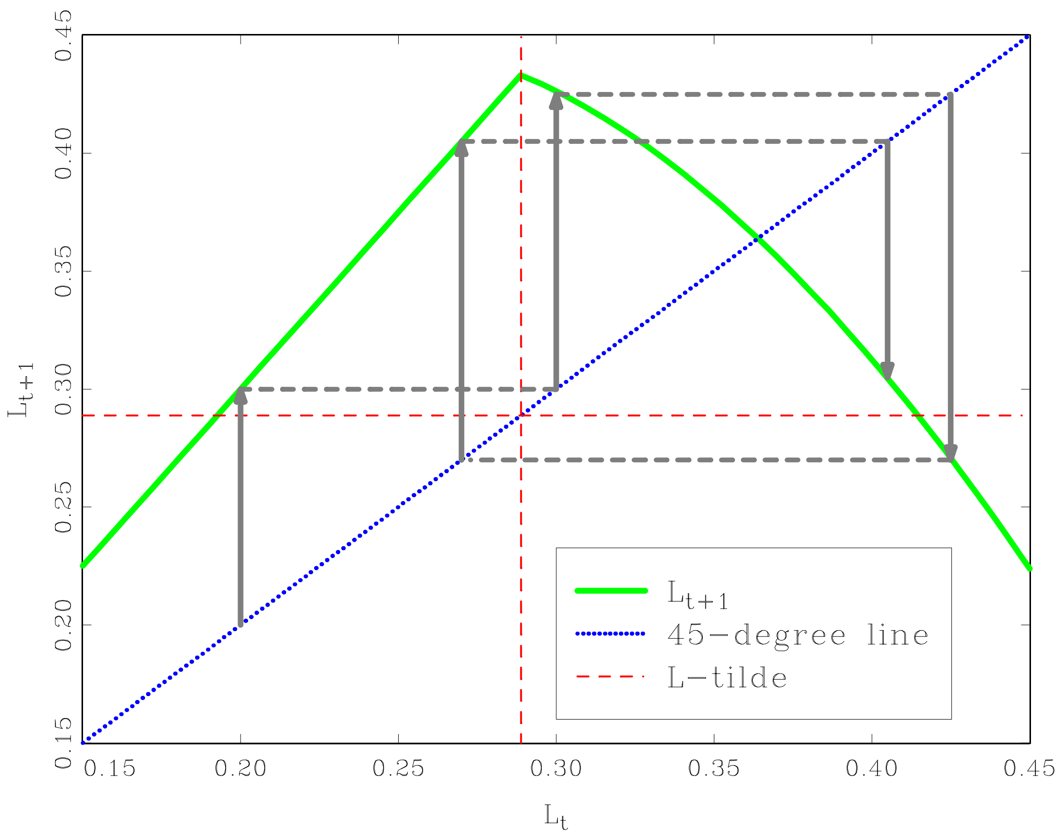


Figure 3: Reduced form population dynamics (baseline parameters).

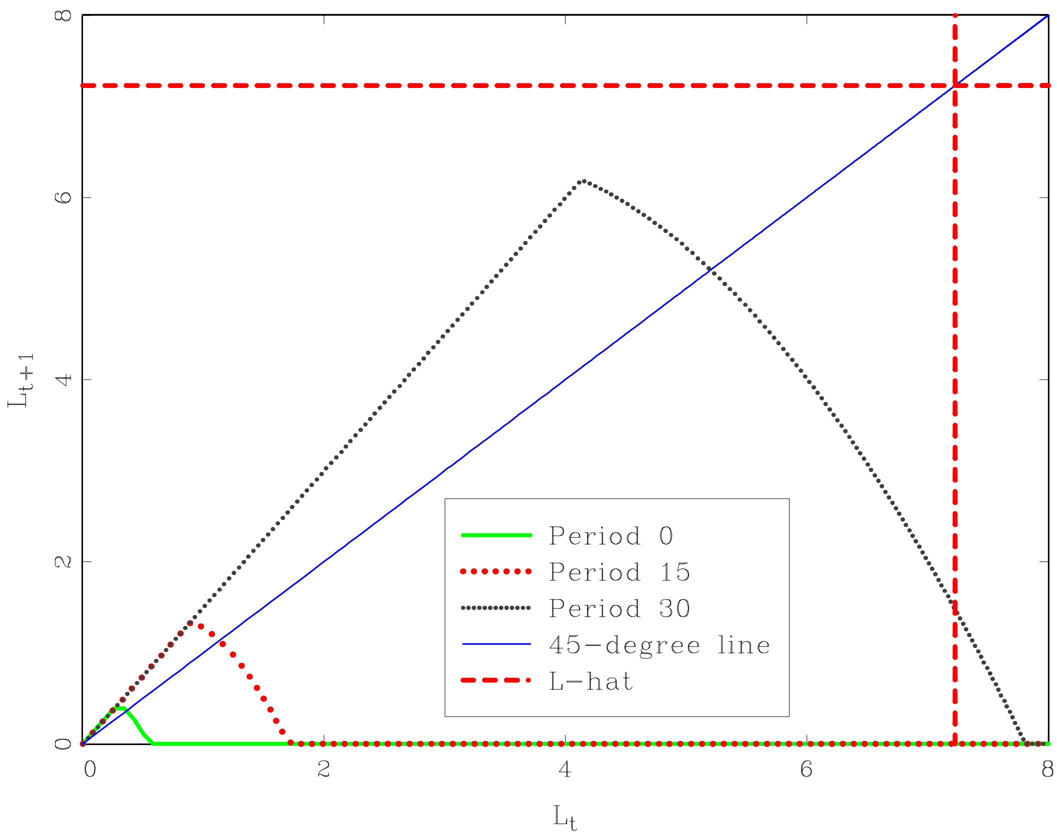


Figure 4: Rough illustration in terms of the reduced form population dynamics of what would happen over time in the full dynamical system. That is, the function mapping $L_t$ to $L_{t+1}$ shifts out over time.

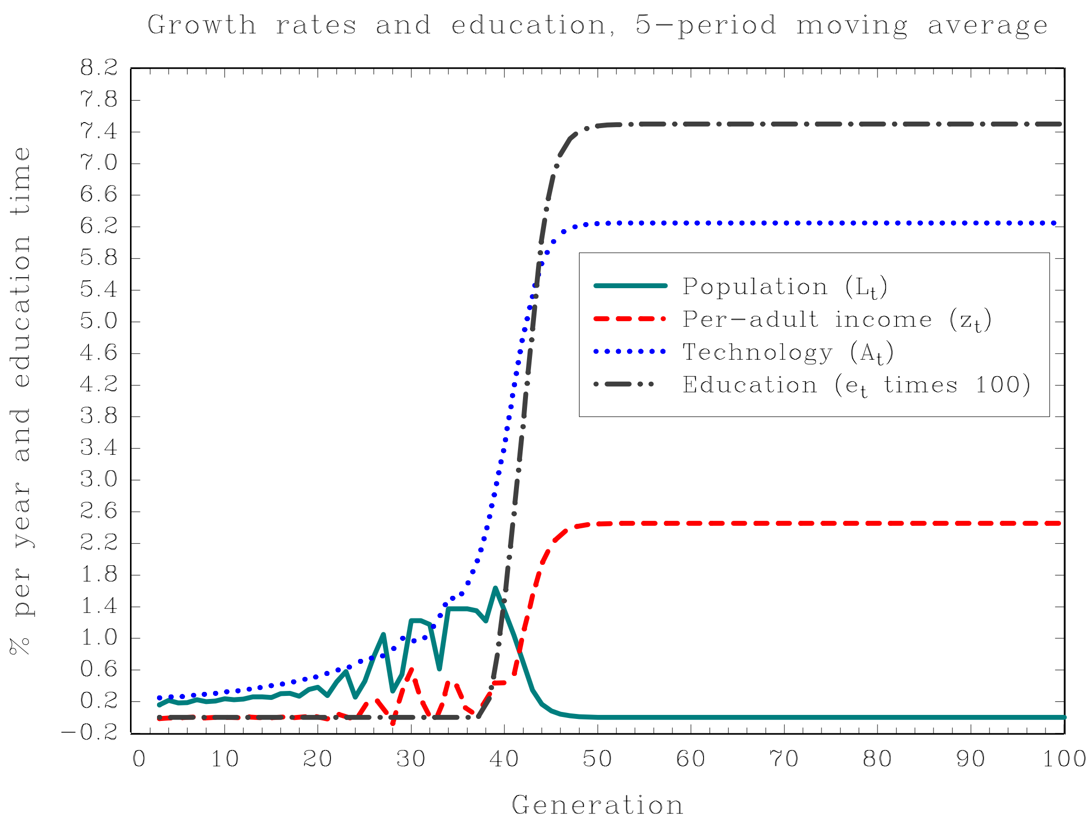


Figure 5: The paths show the level of education time ($100 \times e_t$), and growth rates of all other variables, in the baseline case.

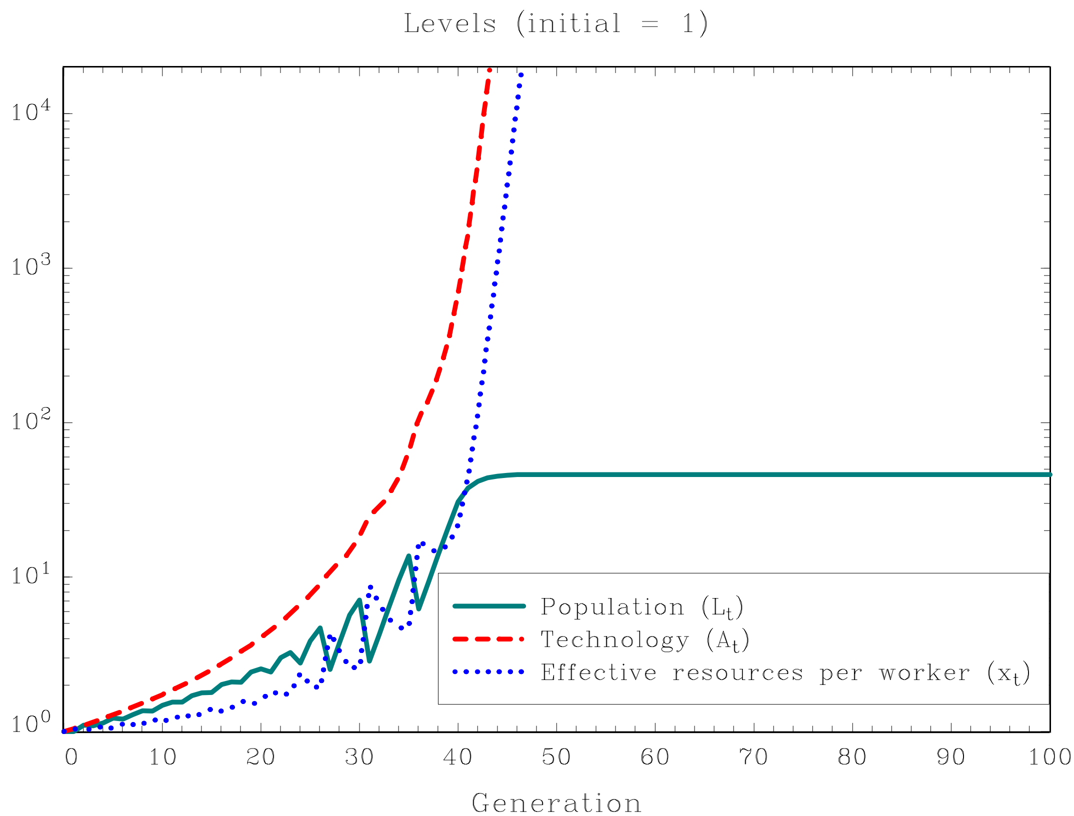


Figure 6: Levels of some variables in the baseline case.

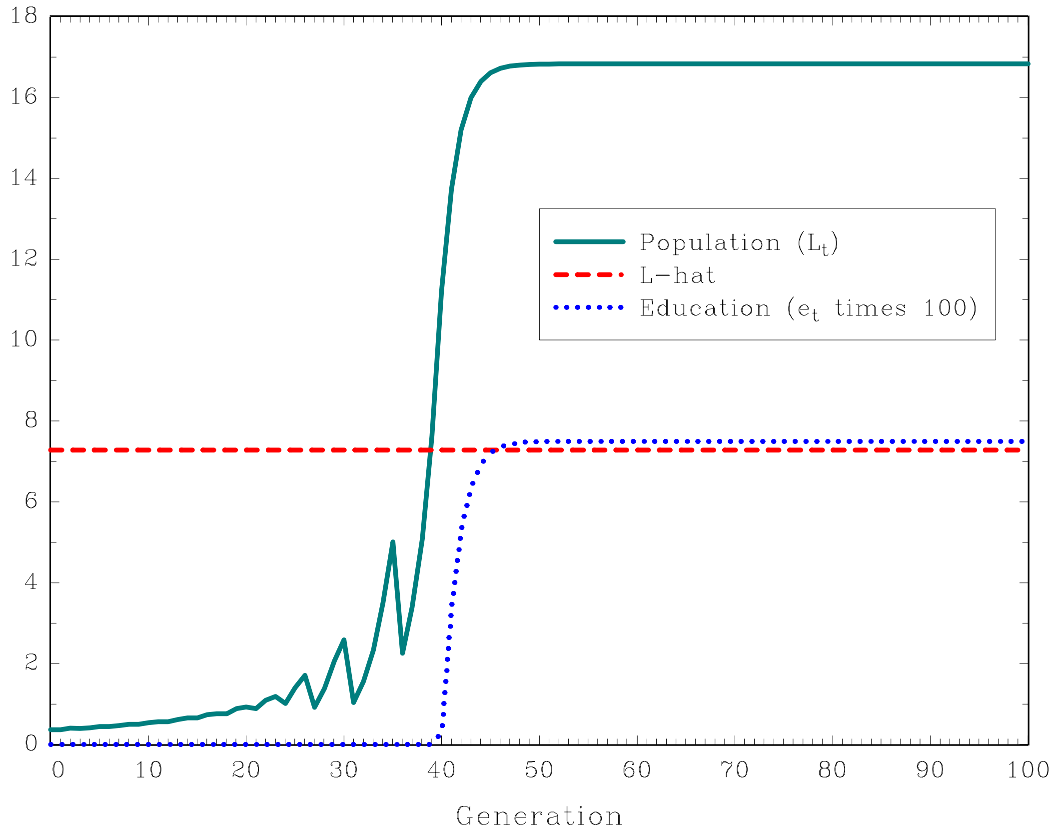


Figure 7: Education starts rising when the level of population reaches the threshold, $\widehat{L}$. (Baseline case.)

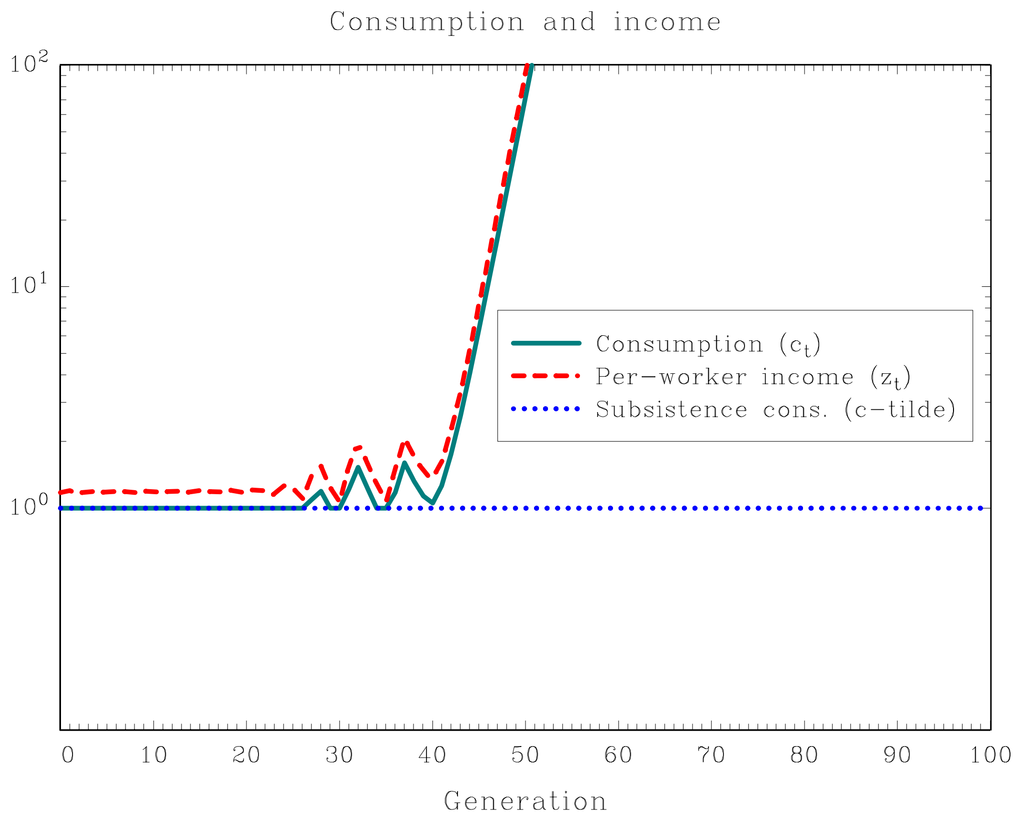


Figure 8: Consumption and per-capita income levels, relative to subsistence consumption, $\widetilde{c}$. (Baseline case.)

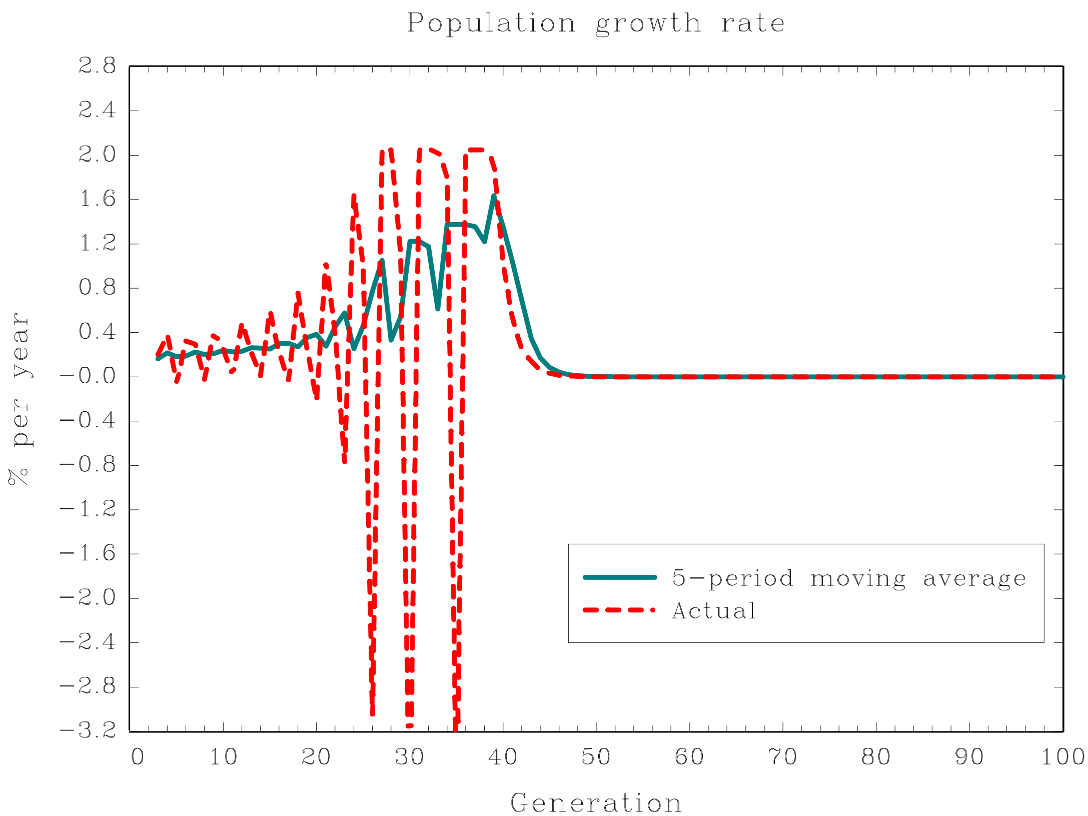


Figure 9: The population growth rate in the baseline case, with and without smoothing.

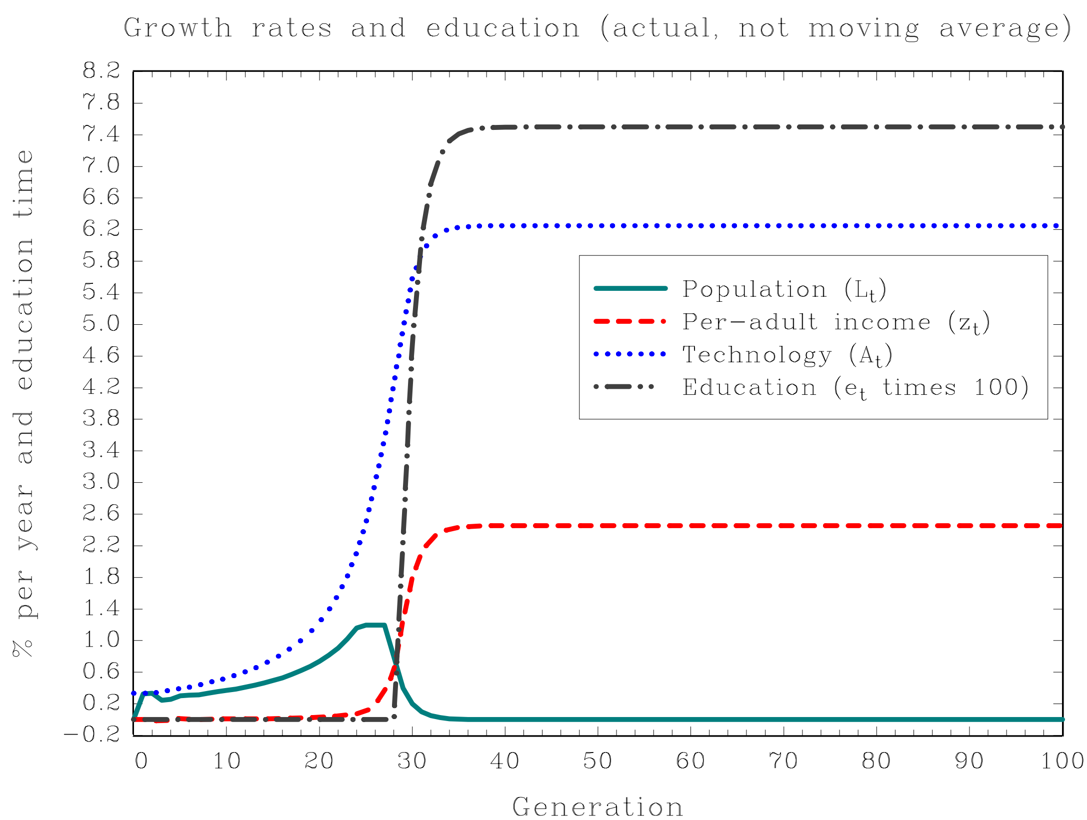


Figure 10: The paths show the level of education time ($100 \times e_t$), and growth rates of all other variables, when setting the fixed time cost, $\tau$, higher.

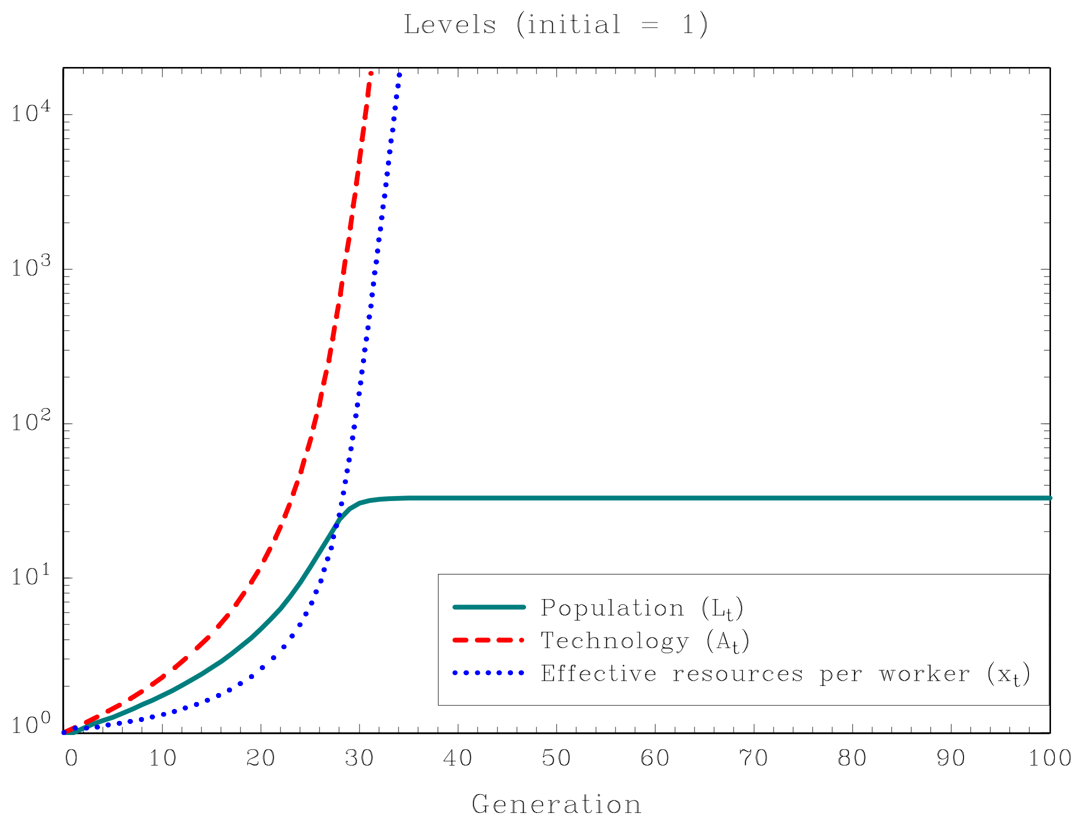


Figure 11: Levels of some variables when setting the fixed time cost, $\tau$, higher.

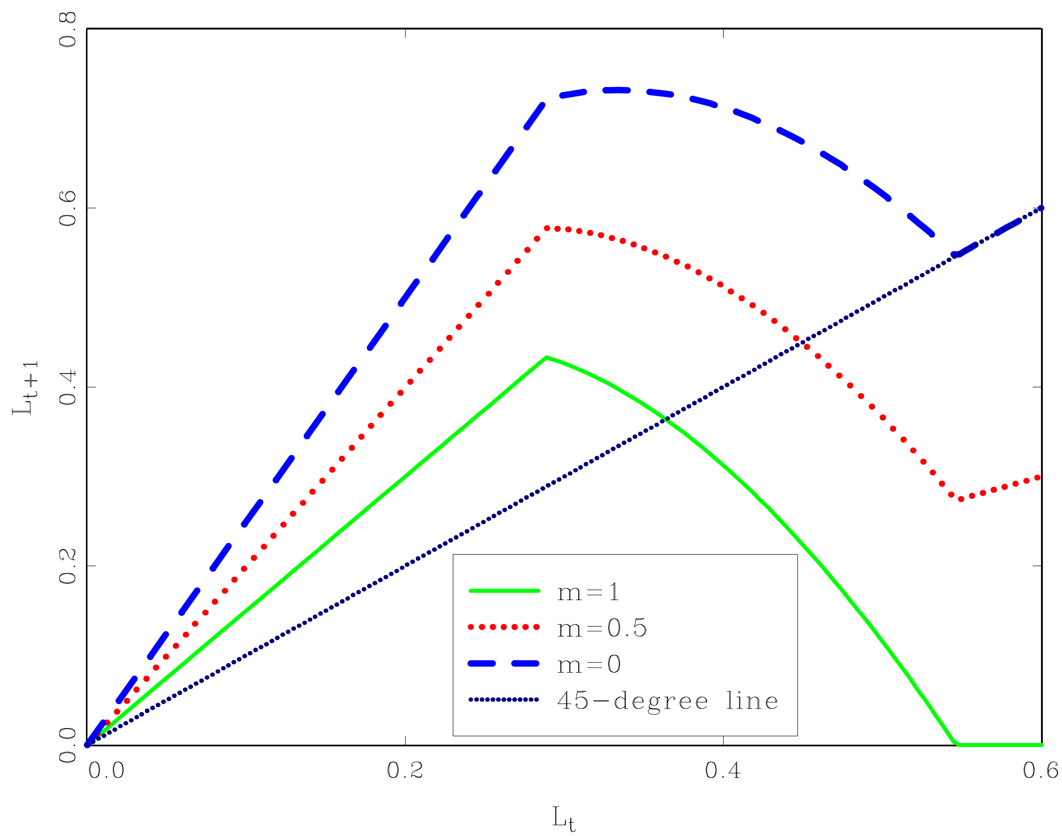


Figure 12: Reduced form population dynamics for different levels of $m$, where all other parameters are held at their baseline values.

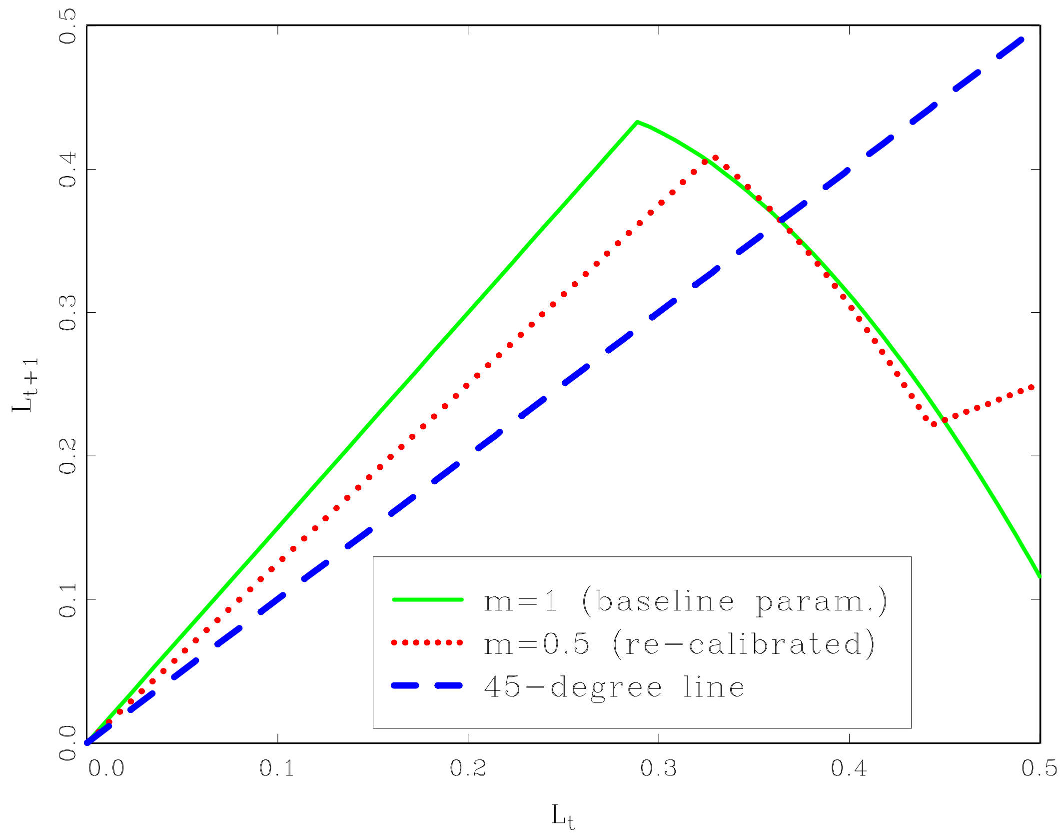


Figure 13: Reduced form population dynamics, where the parameters for $m = 0.5$ have been recalibrated to make initial population the same as in the baseline case.

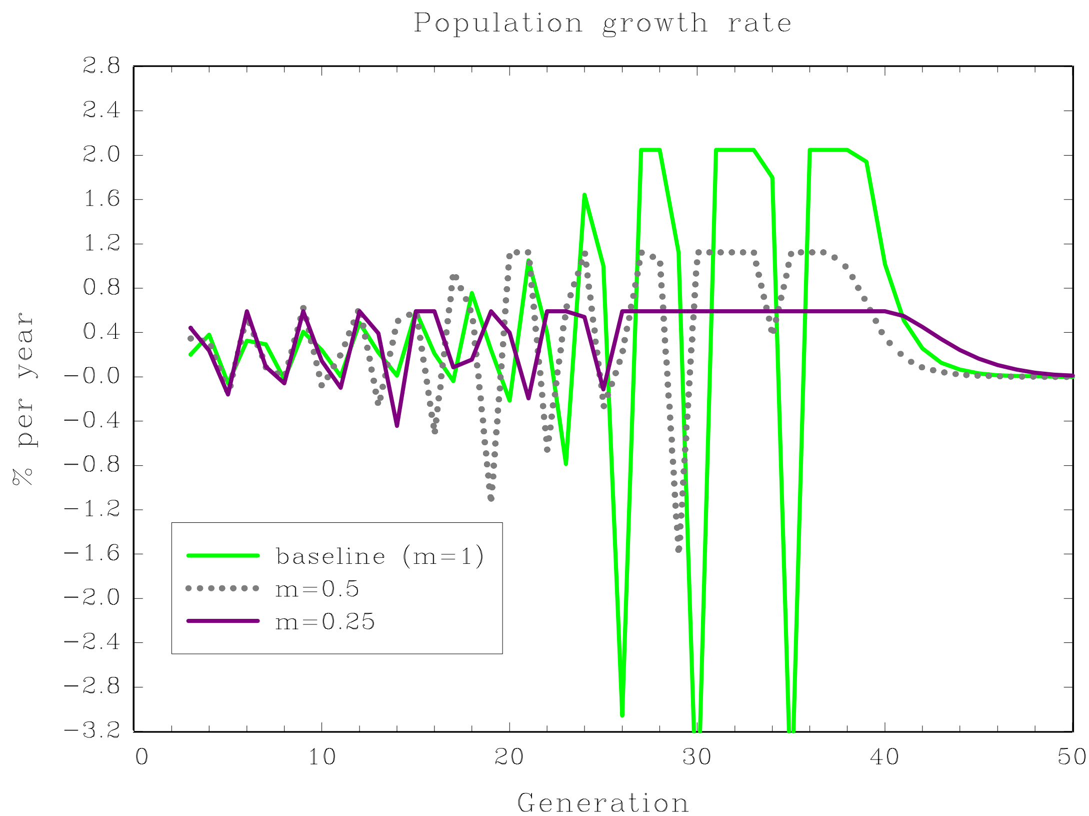


Figure 14: The paths for the population growth rate for different values of $m$, and other parameters recalibrated.

| **Parameters** | **Interpretation** | **Value** |
|---|---|---|
| $\alpha$ | labor share | 0.6 |
| $\tau$ | fixed time cost of children | 0.15 |
| $\rho$ | educational part of $\tau$ | 0.879 |
| $a^*$ | scale effect parameter | 11.42 |
| $\gamma$ | weight on fertility in utility function | 0.225 |
| $\theta$ | scale effect parameter | 1 |
| $X$ | land | 1 |
| $\widetilde{c}$ | subsistence consumption | 1 |
| $m$ | adult mortality | 1 |
| **Endogenous variables** | **Interpretation** | **Value** |
| $e^*$ | Education, modern growth | 0.075 |
| $g^*$ | Techn. growth, modern growth | 2.362 |
| $n^*$ | Fertility, modern growth | 1 |
| $\widehat{L}$ | Threshold population | 7.278 |
| **Initial conditions** | **Interpretation** | **Value** |
| $n_0$ | Initial fertility | 1 |
| $L_0$ | Initial population | 0.364 |
| $A_0$ | Initial technology | 0.870 |
| $e_0$ | Initial education | 0 |
| $g_0$ | Initial techn. growth | 0.048 |
| $z_0$ | Initial per-worker income | 1.176 |

Table 1: Parameter values, baseline case.

| **Recalibrated parameters, intial conditions** | **Baseline** | $\tau = 0.28$ | $m = 0.5$ | $m = 0.25$ |
|---|---|---|---|---|
| $\rho$ | 0.879 | 0.851 | 0.879 | 0.879 |
| $a^*$ | 11.42 | 7.54 | 11.42 | 11.42 |
| $\gamma$ | 0.225 | 0.355 | 0.1125 | 0.05625 |
| $L_0$ | 0.364 | 0.287 | 0.364 | 0.364 |
| $A_0$ | 0.870 | 0.951 | 0.704 | 0.638 |

Table 2: Recalibrated parameters and initial conditions, when different from baseline.